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Mathematics News Letter

SUBSCRIPTION, $1.00 Per Year

Published under auspices of the Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics.

To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

VOL. 4 BATON ROUGE, LA., SEPTEMBER, 1929 NO. 1

LET US RESOLVE

A new session is here and it should bring new visions and new resolves. Vacation time brings with it physical and mental relaxation so that now with the beginning of a new term both teachers and pupils need to take stock of themselves, facing squarely the problems confronting them.

If we are not returning to the classroom with a new vision of enlarged responsibilty to the pupils under our care, with new resolution to guide them and direct them to greater success, to minimize their failures, we are losing a big opportunity. There is no more opportune time than the present. Let us make some good resolutions NOW.—D. M. F.

THE GRAPHIC METHOD OF PROBLEM ANALYSIS

By CARRIE GREHAN
Joseph Kohn High School

We realize that there is a sad lack of power on the part of our grammar school graduates to analyze a new situation and

reason out the processes to apply in solving it. Therefore it is but natural that we wonder where and why we have failed.

Stone contends that the aimless juggling of all figures in a problem, which so frequently takes the place of thought in the solution, is due to one of three things:

1. The child may be subnormal intellectually.
2. He may have been trained with a text book in which there is improper material.
3. His condition may be the result of improper teaching.

It is with the last of these causes that we who would do our best are concerned.

At one time Arithmetic books were filled with catches. I imagine all of us have experienced or heard of the teacher who ruled with such a book. Problems were assigned and some eager parent or big brother thought them out at a great loss of time. Someone's mind received the discipline, but just whose is a question for dispute.

But that is a long buried method we hope. Textbook makers have discarded the descendants of such mathematical curiosities as the problem of Achilles and the hare, and the mule and donkey problem of Euclid. We are now emerging from the day when "social needs" was the only goal, and the text book was an excellent accomplice in teaching arithmetic by the practice method, if I may so call it. Remember how the problems came classified as to process,—the child gave one glance at the top of the page and away he went. Problems could be reduced to a few standard types, and memory on the part of the child was the greatest factor in the solution. Any child could solve them and intelligent thinking was given no chance of developing.

Now we realize a need in Arithmetic as valuable as the utilitarian need—that is the need to develop in the child the power to analyse a new situation and discover the solution. We no longer omit problems because they do not come under one of our stereotyped forms of solution—but today we try to lead the child to see the number relationship and to reason out what to do with such problems, thus training and developing a power to attack and to analyze new situations.

For this reason we have changed our method of attack of problems. Psychologists tell us that economy in learning is a matter of proper habit formation. Then certainly it is a wise

and logical move to teach the child how to attack a problem in a systematic and logical way. The tendency in the past, I think, was either to leave all to the child and by the trial and error method trust to result, or to do too much for him and require no thought on his part.

Two methods of teaching problems have been tried. The first we shall call the conventional method. In this method the child was made to analyze the problem by discovering:

1—What is asked for.

2—What facts are given.

3—How these facts should be used to answer the question.

4—What the answer to the problem is.

The other method is called by John Clark in the April 1925 number of the Mathematics Teacher, the Graphical Method. He uses this name for lack of a better one, quoting the author. But we hope this method will turn out fewer well meaning citizens who like the one in the old story when the minister urged each to give one-tenth of his earnings to the Lord, cried out, "Why only one-tenth? Let's be generous and make it one-twentieth",—or who fails to see the absurdity of a rate of discount well over 300%.

Just what is this graphical method?

Well we start it way down in our primary grades. Thorndike tells us that a problem should preferably deal with a situation that is likely to occur in reality—and the project work of the primary grades, by presenting the actual situation does away with the difficulties of word situation and the child knows:

1—Just what the question is.

2—What facts he must use to answer it.

3—How to use these facts in their right relations.

Real situations are true situations and arouse graphical pictures and with a clear question in mind the child is protected against folly. He actually thinks and decides what to do—and it is to be hoped that such training will eliminate the children who add because there are three numbers in the problem, subtract because there are only two, and multiply because one is large and the other is small.

When the sixth grade is reached we are ready to train the child along a fixed line of thought. We direct him to determine first of all what is to be found in the problem—them to decide

upon what this depends, and what each of these dependents in turn depends upon until he has discovered the essential facts and relationships in the problem.

We try to provide real or at least feasible situations. We encourage the child to identify himself with the person whom the problem represents. We free the problem from difficulties caused by its vocabulary and structure or the lack of experience by the pupils of facts described.

Every sixth grade teacher knows that increase and decrease, cost, selling price, and profit, when met in a problem, mean absolutely nothing to half of her class.

But returning to our graphical method of analysis. Pupils have little or no difficulty in deciding what they are asked to do —so that the first step in our method needs no discussion. When however it comes to discovering upon what this depends, upon correct relationships and how to operate these for a correct solution the difficulties are manifold.

Success in problem solution depends upon the child's understanding of relationships, and study has disclosed the fact that we must lead him to set up associations between the situations and the nature and order of the fundamental operations which make possible the solution.

How then can we help him?

In solving a simple problem two acts of judgment are required—first, the selection of the operation, second, the terms to employ—the former causing the greater difficulty. By requiring the pupils to give a straightforward statement of his answer the teacher will see at one step whether or not the reasoning is at fault. If the child has failed it is usually because he does not understand the situation or the wording is not clear. A great help here is to have the child restate the problem and write down briefly but separately each fact stated and question asked. If the child fails to see the relation between the terms the problem situation must be studied and made real. Here it is often possible to represent the relationship of terms by objects or drawings, encouraging the child to do the drawing himself.

Another help where the relationship seems obscure because the numbers are large, or fractions, or decimals, is to have the child read the problem through and substitute small numbers or read only the integers. This usually gives him a clear un-

derstanding of the operation necessary to answer the question. Roantree and Taylor approve of this method and I can say from experience that it affords a wonderful means of attack for the child who sees only two fractions or two decimals in a problem.

In studying the relationship between the required term and the given term it certainly helps the thinking to express each term in a definite and graphic form. For example—to the average seventh grade child the following problem is a riddle:

"Frank wanted to cut a piece of wire 12 ft. long into two pieces so that one piece would be 1/3 as long as the other. How long must each piece be?"

Much thought might lead to a solution—but once the child learns to draw a few representative lines the situation becomes real and he uses the terms correctly to find the length of each piece and leaves the problem with a satisfied mind.

Imagine what an annoying problem is cleared when the child learns to picture the following situation:

"At a special 1/4-off sale a chair cost $48. Find the regular price." You might get the child to reason out the above situation—but you would never see one half the light of knowledge in his eyes that you will see if you lead him to make a graph of the facts stated and then come to a conclusion. The result will be reasonable and not the result of juggling figures.

We all agree that no set form of so called analysis can help the child solve simple problems and that we do not of course attack all problems this way. Evident problems seldom require teaching and many do not admit of a graphic aid. However we also know that mechanical habits and superficial judgments based upon the sound of the problem or the number of figures it contains will never develop thought or lead to success in solving new problems. The only habit we can give the child that will ever be of service to him is the correct habit of thinking,—which, where problems are concerned resolves itself into an orderly procedure in attacking problems and in making situations real. This is the chief aim of the method of graphical analysis.

This does not mean that we should discard the similarity of oft recurring types. On the contrary general statements of these relations are a real advantage and make possible the solution of other problems without thinking through the problem. By all means fix such facts as: Selling Price is Cost Plus Profit.

Another difficulty in problem solution is the gap from one-step problems to those of more than one step. It is our duty to make the child aware of the new difficulty and then help him to meet it. It is one thing to make the method of solution clear after the problem has been solved—but this is quite different and unrelated to teaching a child how to attack a complex problem filled with difficulties to him. Here again the Graphical Method, called by Roantree and Taylor the Analytic Method, proves its value. Since it cultivates good habits of thought and leads to the desired goal step by step, it is admirably adapted to clear thinking in a complex problem. Suppose the problem is "A boy bought 300 apples at $2.75 per hundred and sold all of them at the rate of 3 for 10c. How much did he make?" Have the child diagram the step, beginning with the required fact, branching to the dependent facts and so on.

$$\text{Profit}\begin{cases}\text{Cost}\begin{cases}\text{Number}\\\text{Price}\end{cases}\\[2ex]\text{S. Price}\begin{cases}\text{Number}\\\text{Price}\end{cases}\end{cases}$$

Just imagine the picture such a line of reasoning will leave in the mind of the child learning to find Rate.

Wherever the problem involves fractions and percentage we turn to the graphic aid and it never fails to prove valuable. Consider such a problem:

"After 24 dishes have been taken from a can of ice cream that was 5/6 full it was still 1/6 full. How many dishes remained?" I could never teach such a problem without an aid. Let the child draw the freezer, mark off the part left and the dishes taken out and almost every child will till you that 6 dishes remain and without having to realize that 1/6 of one thing is 1/4 of another—quite a difficult thing for a child's mind to grasp.

Certainly the child who thinks out the three steps in a problem involving time at increased speed although it takes him some time is better off at the end than the one who has juggled numbers aimlessly for the same length of time.

Today we go a step further in trying to make the reasoning part of a problem uppermost. We give the child problems in which there are no numbers and for which his only solution

is the giving of the steps involved. When he deals with a problem in which there are figures we occasionally make him estimate his result or at least check its reasonableness for himself.

Hand in hand with the graphic aids in teaching children how to attack problems have come the graphs as a means of vivifying facts and relationships which expressed in ordinary form make little appeal to the child and hence lose their real significance.

In conclusion, I think the graphical method of problem analysis is so-called because in this method we so often turn to words, diagrams, lines or graphs to clear the picture and vivify the problem—only as an aid to forming good habits of thinking.

What must I find first? What next? What next? And so on. This is the state of mind with which we wish to have the child attack problems. In other words we try to develop in him the power to see the many simple problems into which some problems may be separated. We try to have him satisfied to do one thing at a time and not think he must understand and solve the problem at once—often before he has even read it through.

We hope such training will develop in our graduates of tomorrow a desire and ability to grasp and analyze logically new situations.

MODES OF STUDYING GEOMETRY

DORA M. FORNO
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Reformers in the field of education early recognized that many teachers had gone astray in the teaching of geometry, as well as in other branches of learning. Its aims and purposes had become entangled. The disciplinary aim had assumed a role of predominating importance which influenced the content in the courses of study and determined largely the point of view and mode of instruction.

The older type of geometry lesson had deteriorated into the formal memorization of definitions and axioms, the demonstration of theorems and problems at the blackboard, and the working out of a few originals by the extra bright pupils of the class. The

proofs of theorems and problems were learned verbatim, even to the placing of the letters in the figures. Remembering the logical sequence of hypothesis and conclusion on the part of a large number of students was not an act of logical thinking but simply an act of memory. The recitation period consisted largely of a repetition of the memorized theorem or problem. Very few students knew what it was all about and only the gifted few profited at all by the study. I feel certain that if we were to make a survey of the schools of this country that we would find some teachers still wearying their pupils with these same hum-drum methods. Yet we know that teachers at large and textbooks writers have had a new vision of what geometry should mean to the individual learner. He should be made to appreciate its immediate or direct usefulness and power in life, he should appreciate the fact that the study of geometry has disciplinary training in developing concepts and reasoning habits by which the quantitative thinking of the world is done, and he should appreciate the beauty of geometrical form to be found in nature, art and industry.

The mode of teaching and the type of text-book will determine largely the degree of interest and success of the student in the study of geometry. It is a well known fact that few subjects of instruction are intrinsically interesting to the young student, but they must be made interesting and that is the job of the teacher if real success is to be attained. It is of significant importance that in the earlier periods of instruction the expressing of facts and methods of attack are more important than logical organization of subject matter. Informal work, frequently called intuitive or experimental geometry, of a constructive character, should precede formal demonstrative work. The writers of modern junior high school texts are endeavoring to supply material of such a constructive type, and the wise teacher is making greater use of the plan of "learning by doing."

The mode of teaching which has adopted this psychological principle of "learning by doing" has been termed the laboratory method. It is the natural method that the investigator uses in arriving at general conclusions, whether they are concepts or principles.

The method of attack is scientific. Experiences are

supplied and necessary materials and tools are at hand for observation and experimentation. The students are directed to make observations and state facts. There must be a careful searching for, selecting and weighing of facts. An active analysis of data at hand must be made before a tentative conclusion can be formulated. A logical demonstration is finally organized and the general proposition is stated.

No expensively equipped laboratory is necessary to teach by this method but the mode of teaching must be informal if you are to seize opportunities created by pupils's abilities and interests. The classroom should be equipped with standard models for use in plane and solid geometry and each student should be equipped with a straight-edge and scale, a protractor, a compass and a supply of drawing paper, both plain and graph.

I will report here a lesson on the lateral surface of a parallelopiped which aroused interest and resulted in getting good results. Each pupil was supplied with a model parallelopiped and a ruler. By actual measurement the lateral surface of each was found. A general discussion followed and a tentative formula was stated. This statement was taken as a proposition and a rigorous proof was organized and drawings made. Each step of the proof was verified as the analysis progressed.

The laboratory method is no new mode of teaching, but it compensates in full measure for any extra thought that may be required in planning one's work according to this plan. A recent text, "A Laboratory Plane Geometry" by Wm. A. Austin of Los Angeles and published by the Scott, Foresman and Company is well worth investigating.

SOME COMMON ERRORS IN ALGEBRA

By W. PAUL WEBBER
Louisiana State University

Experience in teaching mathematics in college reveals to most teachers that there are a few outstanding weaknesses in the freshman aside from his general lack of understanding of mathematical method. Among these may be mentioned uncertainty in the arithmetical processes with fractions, solution of

certain quadratic equations by factoring, rather miscellaneous use of cancellation without understanding why. These errors will best be set forth by examples.

1. In attempting to reduce a fraction to simplest form, the following occurs often (almost constantly) with quite a number of pupils,

$$\frac{4a+bx}{2c+3a} \overset{?}{=} \frac{4+bx}{2c+3},$$ in which are cancelled out the a's.

Also, $$\frac{4+bx}{2c+3} \overset{?}{=} \frac{2+bx}{c+3}$$ in which 2 is cancelled from 4 and $2c$.

There seems to be a lack of understanding in regard to common factors in numerator and denominator, a confusion as to the meaning of factor and of term in an expression. In trying to inculcate the minimum modicum of mathematical reaction in these students the teacher is compelled to go back several years in the pupils' school work to start the correction. This takes time from the regular course, bores the better students, and overloads the unfortunate one. For the latter must struggle to correct his old notions as well as keep up with the new course. This is too much to expect with most of them. One may offer the suggestion that the new course is too difficult. Is that an answer? Does one go to college to do over again that which should have been done earlier? Why not just go through the grades again or through high school and save the expense of college? Allowance of course must be made for lapse of memory in certain things. It is not that type, however which makes the chief difficulty, for these soon get on their feet, so to speak.

II. Another frequently occurring error with the weak students is illustrated by the example,

$\sqrt{25-16} = 5-4 = 1$ or in literal form, $\sqrt{a^2+b^2} = a+b$

The trouble here seems to be in a lack of knowing the signifiance of "square root." The definition of "square root of a number" must be "burned in" so thoroughly that pupils can use it readily and understandingly. It can be done at no "terrible cost." If pupils are made to form the habit of thinking that the square root is one of two like quantities whose product is the quantity

whose square root is desired, the error should be materially reduced.

III. In solving quadratic equations by factoring such error as the following frequently occurs.

$x^2-6x=9$

$x(x-6)=9$

$x=9$ and $x-6=0$. So that $x=6$.

The error lies in the fact that not all terms are put on one side before factoring or, else, the factor taken out is a factor of one side only. This distinction should not be beyond the reach of any one aspiring to higher education. Possibly more attention should be given to the nature of factoring and to the fact that a product of two (or several factors) is zero when and only when at least one factor is zero.

It is better to make certain a limited number of fundamentals than to cover a broad course and make nothing certain. The writer would prefer that his new freshmen should know the fundamental processes of arithmetic so well that they can not be fooled in them and to know how to apply them to algebraic expressions of average difficulty, rather than to be half proficient in the whole prescribed text. Some of the time used in factoring pages of such as $x^{19}-y^{19}$ and $a^{21}+z^{21}$ might profitably be used to more firmly fix the fundamentals with a somewhat simpler set of exercises and leave a little more time to make fractions certain.

Some of these questions were discussed in the writer's class in "The Teaching of Secondary Mathematics" during the present summer. It was brought out that, with many teachers, a suggestion may be all that is necessary. But with those teachers, of, say English or History, who must (often against their wills) undertake a class in algebra or geometry, "to help out," the difficulty is more serious. The ultimate outcome may be sad for the pupils. There is a responsibility which the schools owe the pupils, and it is desirable that all should be given the best possible opportunity. At any rate, all can consciously try to make the existing system more and more efficient.

The writer believes that the News Letter would welcome suggestions of ways of reducing the number of instances in con-

nection with the errors cited above and of others that may need attention.

Finally, in testing for proficiency in any mathematical operations, there should ultimately be a test on combinations of the operations in a wide variety of circumstances.

MAKING GEOMETRY WORTH WHILE

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So highly specialized has modern industry become that large numbers of us are in danger of ceasing to feel the community of human interests. This is true in spite of the fact that every industrial group is becoming increasingly dependent upon other groups. A high degree of specialization insures the finished product, but the specialist who turns it out sacrifices an interest in common with you and me. If the day ever comes when tax-supported schools and colleges are dominated by professional and vocational interests, from that day forward the saving grace of civilization will be largely in the church and private institutions.

Never before, as now, has there been so crying a need for education in the fundamentals, for the sort of education that secures to the individual a fine possession, not of specialized, but of normal powers of mind, feeling, taste.

It is the age-old matter of the humanities versus the utilities. If I spend the receptive years of life in merely developing a trade so that earning power may be maximum, I restrict to a very narrow range the idealism and the capacity with which I may be endowed. It is infinitely more worth-while to build the highest life than to make the best living. If this principle is rejected from the underlying basis of training for life that training is fundamentally wrong. We are not to be misunderstood. Material livings must be made, and it is both legitimate and necessary that our skill in earning them be sharpened—but it is not necessary to compass the goal of increased skill by dispensing with the higher activities of soul and mind.

Geometry may be used to compute the acreage of a field or

the contents of a tank but such practical uses are not sacrificed by concomitant uses of it as an instrument of discipline for processes which can be made to figure effectively in every problem-solving task of any field whatsoever. To quote from a previous monograph by the speaker: "Subject to re-use in other fields (than geometry) is the process that arranges, classifies and accurately values the items of an hypothesis, that notes the immediate consequences of these, and carefully provides for each item a train of inference to connect with the main line of the argument. * * * * Fixed into a habit by sufficient repetition, the process functions under any situation that calls for a problem to be solved or a complicated question to be settled." "Induction, deduction, observation analysis, synthesis, experiment are no more legitimate as instruments for discovering the laws that bind together the facts of botany, or of geology, than they are for finding the relations that tie a body of assumptions in geometry to their logical implications."

Geometry material is ideally adapted to all the exercises that engage the mind when it correctly attacks any problem in any field whatsoever. We say "correctly attacks," because too often are problems of industry, public service and other fields attacked incorrectly, or inadequately, by those whose task is to solve them. Ability to do straight and correct thinking is infinitely more to be valued than ability to draw a straight line or to compute acreage. Not all who pass from the schools have occasion to compute acreage. But all who pass from the schools will be confronted with problems—and grave ones—however widely diverse the fields in which they lie. To the extent that common processes and methods are used in the solutions of these problems is there a common interest binding us all together. Moreover, if the same fundamental activities of mind that are used in solving the geometry problem are used again in attacking the industrial problem, it follows that the engaging of these activities in geometry has been a training for their application to the industrial problem. The same truth will hold for any type of problem whatever.

We assume what we believe is granted by psychologists that, to the extent that a process of mind is repeated, to that extent does the process tend to become a habit. We mean by process a

succession of mental acts, or states, whether simple or complex, it matters not.

Consider an outstanding mark of the geometry of Euclid, namely, the practice of setting up a formal statement of the hypothesis of each theorem: "If two circles intersect each other." "If three or more parallels intercept equal parts on one transversal." "Given two diagonals of a rhombus." "Given a triangle." "Given a regular polyhedron." "Given" and "if." They compel from the mind an act, or a sequence of acts, of attention to a definite object, or set of objects, with accurately defined properties. These properties furnish a key to the door, or doors, opening upon a new vision of geometric relations. Since the key to this door is discoverable only in these properties, a searching quality of mind must be used to find it. Examination of the items of the hypothesis of a theorem calls ordinarily for but little labor of mind, but little as it may be, it snould be none the less thorough, one in which are employed exactness and completeness of observation. How often has a teacher seen even the bright student lose his way because he overlooked some item of his data!

We offer some examples of thorough data-mapping processes, and at the same time point out liabilities to error in using hypothesis material. The problem is, to show that "the median of a trapezoid passes through the middle points of the two diagonals."

Items of hypothesis:

1. A trapezoid.
2. The median of the trapezoid.
3. The two diagonals of the trapezoid.
4. The middle points of the diagonals.

Inattention to this delineation might lead to one or more of the following errors:

(a) Using an isosceles trapezoid instead of a general one.

(b) Errors from a failure to classify the mid-points of the diagonals as a part of the hypothesis.

(c) Omitting some relevant necessary implication of the definitions of trapezoid and median of trapezoid.

Consider the problem: "If two straight lines are drawn through any point in a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides lie on

the circumference of a circle whose centre is the point of the intersection of the diagonals."

Items of hypothesis:

1. A square.
2. The diagonals of the square.
3. Any point of either diagonal.
4. Two straight lines through this (any) point, one parallel to two sides of the square, the other parallel to the remaining sides.
5. The points of intersection of these lines with the sides of the square.
6. The point of intersection of the diagonals of the square.

Inattention to this delineation might lead to one or more of the following errors:

(a) A mistake as to the part in the proof played by the circle mentioned in the general statement.

(b) Unwarranted extension of the hypothesis to include the case when the "any" point is not on the diagonal but on an extension of the diagonal, that is, outside of the square.

(c) Omitting some relevant necessary implication of the definitions of square, circle, parallel.

We do not need to say that this sort of elaboration of the hypothesis material is often unnecessary, especially when the problem is manifestly easy. But when such is not the case a data-map is remarkably effective in reducing the difficulties of solution.

The habit of mapping and examining the hypothesis elements of a problem may be thoroughly acquired, imperfectly acquired, or acquired not at all. Manifestly it is absurd to discuss the transfer of habits from one problem field to another when there are no habits to be transferred. Responsibility for the formation of a "data-examining habit" in the student of geometry must be jointly shared by teacher and student. Certainly the geometry material within itself has no more power to bring about such a development than a stone lying on the river's brink has to create a geologist.

Will the thoroughly formed habit of examining the hypothesis elements "carry over" into all other problem fields? The answer is, nothing of itself will "carry over" until it is carried.

Habit, like everything else, is effective in proportion as its control and direction are in accord with proper attitude and intelligence. The habit of walking fifteen blocks before breakfast is not a good thing on the days when the mornings are stormy. The habit of going to my class-room every morning by walking west six blocks and then north five blocks, is, for a time, a good thing if, previously, I have been taking no physical exercise. The same habit develops a bad side if, for a period of ten years I walk to my classroom in no other way. The habit, even when it has been completely formed, of examining with scientific thoroughness the data of every complex geometry problem is subject to effective, ineffective or indifferent use according to the purposes and attitudes that have been allowed to control in the formation of the habit. If while forming the habit I have steadily nursed the conception that such habit will be utterly valueless outside of the geometry field, naturally I have closed the door to the building up of another habit to which I now refer. This is the habit of undertaking to find in the data of every non-geometric problem an opportunity for re-applying the mental processes that have been made to operate on geometry data. On the other hand, if while forming the habit in geometry, it has been my purpose as a student, with the help of the teacher, constantly, as far as possible to be vigilant to note opportunities for re-applying to every type of problem the same mental processes that are being made to function in the geometry problem, then will the fruitage of geometry discipline be in a certain way to be transferred to other problem fields.

At this stage of our study a remark should be made with a maximum of emphasis. It is this: What has just been said is as true of all the other mind-process-habits that may be formed in using geometry material as it is of those associated with the use of hypothesis material.

We now proceed to the proposition that to the extent that a problem is inexact, or contains indeterminate elements, to the same extent, and even more, should care be exercised in examining the data on which the problem is presumed to be based. To search within a relatively limited body of assumptions for the key to paths leading to a geometric structure must, ordinarily, be far less taxing on attention and perception than to search through

a much wider and less definite range of data for a key which, when found, may not fit the desired door. If, then, attention and concentration are necessary to prevent oversight of fact, or implication, in geometry premises, much more are they in demand in preventing oversights of fact, or implication, in the wider fields of less exact data.

Another characteristic of Euclid's geometry is the formal setting forth of the conclusion to be reached, or the objective to be established before the deductive process is begun. Given a point and a straight line. Prove that between the two but one perpendicular can be drawn. How often is the bewildered student set on a correct path by being required to re-state his objective; nor is an occasional glance at his goal always sufficient, but, frequently is strong advantage gained if he keeps it in mind steadily. What bracing, strength-giving influence issues from a sense of purpose! What a difference it makes even in a deductive process! Let a vigorous young mind be faced with two propositions, one of them being "Given a circle, to deduce from its definition all you can by the rules of logic," the other being, "Given a circle, to deduce from its definition that equal chords are equally distant from its centre." How different will be the reactions of such a mind to the two problems! The latter is like a challenge to take a message to Garcia! The former issues no challenge, but pictures a desert of monotonous deduction! Mathematical history is full of testimony to this truth. The non-euclidean geometries that have sprung up in the last century-and-a-half would, possibly, have not been born had it not been for centuries of the pursuit of a definite objective by mathematicians, namely, a proof of Euclid's postulate of parallels. Newton sought for an easy method of quadrature and discovered the binomial theorem. Search for Fermat's lost solution of $X^n+Y^n=Z^n$, n positive, integral and greater than 2, has resulted in a wonderfully expanded number theory, rich in the discoveries of Euler, Kummer, Dickson.Leibnitz, aiming for the discovery of a method of elimination between linear equations, laid the foundation for the beautiful theory of determinants.Thus almost indefinitely might evidences be cited illustrating the stimulating effect upon research of the process which we may describe as a conscious mental-focusing upon a specific objective.

Adapted, with proper change of terms, to what we may dare

to call this focus-upon-an-objective process of mind, the fundamental principle above applied to the hypothesis-examining process, may be re-stated: If in any study of geometry problems I have developed a habit of being consciously guided by the demands of the objective, the "quod erat demonstrandum," and, if side by side with this habit, I have also constantly endeavored to build up another habit, namely, the habit of endeavoring to unravel problem difficulties in all other fields by applying to these difficulties the same focus-upon-an-object process of mind that I have used in my geometry, then, manifestly, I have consciously, deliberately, carried over and made to function in such fields a process used in my geometry.

Before passing to consider a third mark of geometry processes we list some examples of actual, or possible, application of the focus-upon-an-object process in non-geometric fields.

(1) The commander of an army usually develops a victory over the enemy through a succession of achieved objects, for instance, by attacking and capturing successively, several strategic points.

(2) A federation of labor organizations has a strong sense of its objectives—however little, or much, it may have been whetted by geometry.For this cause the members of a labor union usually act in most marked concert.

(3) A political party has permanent coherence and existence largely because it sets forth and battles for the realization of definite objectives in the expressed life of the nation or state.

(4) A city's executive with a highly developed sense of objective and with the impulse to make it function, will, without waiting or wavering, determine upon policies consistent with the city charter (the hypothesis) and immediately proceed to their execution.

Formal statements of the hypothesis and of the objective to be reached by starting from the hypothesis are characteristic of Euclid's organization of geometry material. Other marks, not only of Euclid's geometry, but of all other geometries, are so general in their character that they may be said to belong to every type of mathematical process whatsoever. One of these marks may be described as follows: When the objective of a proving process has been reached, the mind is logically certain of

the fact; on the other hand, when the thing to be proved has not been established, the mind is logically sure of this also.

One's certainty that every angle inscribed within a semicircle is a right angle is arrived at by the same general process of mind as that by which one reaches the certainty that two atoms of hydrogen combined with one of oxygen form the substance called water. On the one hand the mind traces the consequences of rigid laws of circle, on the other hand it traces the consequences of rigid laws of chemical affinity. It is the same process that the mind is put through in reaching certainty of conclusion in physics, or in any other exact science.

Is this process applicable in all problem fields? If while constantly submitting my mind to such a process in the problems of geometry I have endeavored constantly, with the aid of the teacher, to apply to all other types of problem the same process, that is, to arrive at solutions of these problems by logically developing them from the inner necessities of their data, as a rose, and only a rose, is unfolded from a rose seed, as a chemical compound is pre-determined by the nature and proportion of the atoms entering the combination, then, manifestly, not only has the process been made to function in other problem fields than geometry, but—and this is of the greatest significance—the use of scientific method in handling the data of every kind of problem, exact or inexact, has been made a habit.

Another group of processes is the group engaged in searching for paths of inference running from the data of the problem to the objective that has been set up. In this group are found all the types of mind action that may be exercised when the mind seriously engages any sort of problem, whether classified as exact or inexact. We say "may be exercised," for it is to be deplored that neither in the mathematical nor in the non-mathematical problem work of today is this group of processes exercised to the extent to which its value entitles it to be exercised. While mathematics may be defined as the science of necessary conclusions, the rigid character implied for it by this definition only indicates its fitness as an instrument for discipling a group of mind activities which must be exercised in all problem work if scientific solutions of the problems are sought. Abandonment of inexact methods for exact ones must be conceded to be a step forward in every kind of problem-solving effort.

The processes by which a mind secures passage from a set of hypotheses to a desired conclusion, or from a body of data to a pre-determined objective are various in kind. To select out of the numerous trains of implication issuing from a given hypothesis one promising to arrive at the objective, demands judgment, judgment demands comparison, comparison, demands attention. The train selected might lead to the goal desired by too wide a detour. Then experiment, more careful induction of facts, the consideration of the probable, analysis of the objective to determine avenues entering it—all these may be needed processes. The mind may have to work its way through a forest of interlacing implications with the same tentative inquiry, wide-awake searching, activity of observation and imagination, balancing of prohabilities and all the discriminations and concentrations that are demanded, but by no means always supplied, in the solutions of the problems of "real life," so-called.

A THEOREM ON PROPORTION WITH SOME APPLICATIONS

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As ordinarily defined the proportion

(1) $$a_1 : b_1 : c_1 = a_2 : b_2 : c_2$$

has no meaning when any of the letters involved have the value zero. Yet the proportion is used in situations in which some of the letters involved may be zero. It is the object of this note to give a definition of (1) which does not exclude the vanishing of some or all of the letters involved, state and prove a theorem based on this definition, and show the usefulness of the result by some applications.

We lay down the following

Definition. The proportion (1) holds if, and only if, there are members r_1, r_2, not both zero, such that

(2) $$r_1 a_1 = r_2 a_2, \quad r_1 b_1 = r_2 b_2, r_1 c_1 = r_2 c_2.$$

Thus we have $3 : 9 : 21 = 5 : 15 : 35$ $(r_1 = 5, r_2 = 3)$, $3:7:0 = 6:14:0$ $(r_1 = 2, r_2 = 1)$, $1:2:7 = 0:0:0$ $(r_1 = 0, r_2 = 1)$.

We now prove the

Theorem. The proportion (1) holds if, and only if,

(3) $$\begin{Bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Bmatrix}^2=0$$

where

(4) $$\begin{Bmatrix} a_1 & b_1 c_1 \\ a_2 & b_2 c_2 \end{Bmatrix}^2=(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2.$$

Proof: Suppose (1) holds. Then there are numbers r_1, r_2 not both zero such that (2) holds. Suppose r_1 is not equal to 0. Then $a_1=(r_2/r_1)a_2$, $b_1=(r_2/r_1)b_2$, $c_1=(r_2/r_1)c_2$, from which (3) follows by aid of (4).

Suppose (3) holds. Then by (4),

(5) $a_1b_2-a_2b_1=0,$

(6) $b_1c_2-b_2c_1=0,$

(7) $c_1a_2-c_2a_1=0.$.

Now if a_1, b_1, c_1, a_2, b_2, c_2 are all zero (1) holds with $r_1=r_2=1$. If either a_1 or a_2 is different from zero, (1) holds by (5), (7) for $r_1=a_2$, $r_2=a_1$. If either b_1 or b_2 is different from zero, (1) holds by (5), (6) for $r_1=b_2$, $r_2=b_1$. If either c_1 or c_2 is different from zero, (1) holds by (6), (7) for $r_1=c_2$, $r_2=c_1$. Hence (1) follows in every case.

From the algebraic identity

(8) $$\begin{Bmatrix} a_1b_1c\, c_1 \\ a_2b_2c\, c_2 \end{Bmatrix}^2=(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2,$$

known as the **Lagrange identity,** follows the

Corollary. The proportion (1) holds if, and only if,

$(a_1^2+b_1^2+c_1^2)(a_2^2+b_2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2=0.$

Applications to solid analytics. From the Law of Cosines of trigonometry and the distance formula of analytics, it follows that if P_0, P_1, P_2 are the points

$(x_0y_0z_0)$, $(x_0+h_1, y_0+k_1, z_0+l_1)$, $(x_0+h_2, y_0+k_2, z_0+l_2)$, respectively, then

(9) $\cos P_1P_0P_2=(h_1h_2+k_2+l_1l_2)/(d_1d_2)$

where $d_1=\sqrt{h_1^2+k_1^2+l_1^2}$, $d_2=\sqrt{h_2^2+k_2^2+l_2^2}$. Also from the formula

$\sin A=\sqrt{1-\cos^2 A}$

and (8), (9) follows

(10) $$\sin P_1P_0P_2=\sqrt{\begin{Bmatrix} k_1k_1 & l_1 \\ k_2k_2 & l_2 \end{Bmatrix}^2}\,/(d_1d_2).$$

Now let P_0, P_1 denote distinct points. Then a point P is on line P_0P_1 if, and only if, either $P=P_0$ or $\sin PP_0P_1=0$, that is,

by (10), if, and only if,

$$(11) \qquad \begin{Bmatrix} x-x_0, & y-y_0, & z-z_0 \\ h & k & l \end{Bmatrix}^2=0$$

where $(x_0y_0z_0)$, $(x_0+h_1y_0+k_1z_0+l)$, (xyz) are the coordniates of P_0, P_1, P respectively. Hence (11) is the equation of the line determined by two points P_0, P.

But (11) is by the theorem equivalent to

$$(12) \qquad x-x_0 : y-y_0 : z-z_0=h : k : l,$$

which is a second from of the equation of the line. Finally (12) holds, by definition, if and only if, there is a pair numbers of r_1, r_2 not both zero such that

$$r_1(x-x_0)=r_2h,$$
$$r_1(y-y_0)=r_2k,$$
$$r_1(z-z_0)=r_2l.$$

Here r_1 is not equal to 0 since h_1k_1l are not all zero. Setting $t=r_2/r_1$, we see that (12) holds, if, and only if, there is *a* t such that

$$(13) \qquad x=x_0+ht_1 \quad y=y_0+kt, \; z=z_0+lt,$$

which are parametric equations of the line P_0P_1.

ON THE TRISECTION OF THE GENERAL ANGLE

W. PAUL WEBBER

This discussion makes no claim to completeness, but may be suggestive of the nature of the problem. The only constructions possible with straight edge and compasses are such as come from,

(a) Intersections of straight lines.

(b) Intersections of straight lines with circles

(c) Intersections of circles.

The equation of any straight line may be put in the form $y=mx+b$, where x, y are the ordinary rectangular coordinates of points on the line and m, b constants. (See Analytic Geometry). Let two intersecting straight lines have for their equation, respectively,

$$y=mx+b \text{ and } y=m'x+b'$$

Solving these by ordinary algebra gives

$$x=\frac{b'+b}{m'-m} \text{ and } y=\frac{m'b-mb'}{m'+m}$$

These expressions can be constructed by means of similar triangles when m, m', b, b' are known. (See Texts on Elementary Geometry.)

Let $y=mx+b$ be the equation of a straight line and $(x-k)^2+(y-l)^2=r^2$ the equation of a circle of radius r and at the point (k, l). Eliminating y by substituting from the first into the second of these equations gives

$$(x-k)^2+(mx+b-l)^2=r^2$$

This is a quadratic equation and can be reduced to form $ax^2+bx+c=0$. Solving for x gives

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

and a similar expression is obtained for y by substituting in either of the original equations.

Such expressions involve only rational operations and square roots. These can be constructed with straight edge and compasses. (See texts on Elementary Geometry, for example Wentworth)

Let $(x-k)^2+(y-l)^2=r^2$ and $(x-k')^2+(y-l')^2=r_1^2$ be the equations of two intersecting circles. The coordinates of the points of intersection are obtained by solving the two equations simultaneously. It is easier to expand the equations in full and by subtracting obtain the equation of their common chord which will be of the first degree. Combining this equation with that of either circle will determine the coordinates. Thus the case of two interesecting circles is reduced to the that of a straight line and a circle.

Only numbers of the form $p+\sqrt{q}$ or combinations of such can come from the solution of problems involving the intersections of circles, or of straight lines with circles.

Suppose the value of x to be found from an equation with rational coefficients is made up of the sums, products and quotients of such numbers as $p+\sqrt{q}$. Write

$$x=f(p+\sqrt{q})$$

a rational function of the argument, $p+\sqrt{q}$. To obtain an equation with rational coefficients having this function as a root it will be necessary to square the above equation one or more times. Conversely the equation might be solved by means of a succession of square roots. Obviously, in rationalizing the above equa-

tion the result will in general be an equation of even degree in x. It may be inferred (It can be proved) that any problem solvable by straight edge and compasses will be an equation of even degree when reduced to lowest rational form. If it cannot be reduced by factoring rationally it can only be solved by operations among which are square roots. If the value of x is of the form above the lowest degree rational equation obtainable in any case will be a quadratic. It may be inferred that if a geometric problem reduces to the solution of an equation which when in the lowest possible rational form is of odd degree the solution is not possible with straight edge and compasses.

To show that the trisection of the angle, in general, is impossible by straight edge and compasses it will be sufficient to show that it does not lead to an equation which in its lowest rational form is of even degree. Write $\cos 3x = 4 \cos^3 x - 3 \cos x$.

This equation cannot be reduced to a simpler rational form. Hence it cannot be solved by straight edge and compasses except for certain special values of x. Let $x = 40°$. Then $3x = 120°$. Subustituting in the equation gives

$$4 \cos^3 40° - 3 \cos 40° + \tfrac{1}{2} = 0.$$

Setting $y = 2 \cos 40°$ this reduces to

$$y^3 - 3y + 1 = 0$$

This equation is of odd degree and cannot be reduced to lower degree, rationally. Hence it cannot be solved by straight edge and compasses.

The angles 180°, 190° and multiples of these can be trisected by straight edge and compasses. For angles 60°, 30° can be constructed.

A construction purporting to trisect angles in general by geometric construction was sent to this magazine some months ago. The writer of this article carried out the construction carefully on several angles. It was clear in every case that a good draftsman could detect the error in the size of the trisected parts. In the case of trisecting a 100° angle the most careful construction led to an error of nearly 2°. The method worked perfectly on the 180° angle for the reason stated above.

Professor Dickson gives a careful discussion of this and other classic problems in Young's Monographs of Modern Mathematics, Longmans Green & Co.